Faculty Working Papers

PORTFOLIO PERFORMANCE, RESIDUAL ANALYSIS AND CAPITAL ASSET PRICING MODEL TESTS

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#539

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Summary:

Recent work by Richard Roll has challenged the worth of portfolio performance measures based on the capital asset pricing model. This paper demonstrates that Roll's conclusions are due to his inappropriate use of a "truly" ex-ante efficient index. Using a choice and information theoretic framework, an appropriate index is shown to be efficient relative to the probabilities assessed by the "market." Residual analyses and portfolio performance tests, using such an index, yield meaningful results for a wide class of information structures. Roll's primary criticisms, however relate to tests of the model itself. We argue that these criticisms are vastly overstated.

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PORTFOLIO PERFORMANCE, RESIDUAL ANALYSIS AND CAPITAL ASSET PRICING MODEL TESTS*

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Edward M. Rice

I. INTRODUCTION

Richard Roll's [1977] criticism of the capital asset pricing model presents an exciting intellectual challenge. We have found especially challenging his conclusions concerning the use of the security market line as a benchmark for asset or portfolio performance and for residual analysis. These conclusions do not depend on the asset pricing model being invalid. Nor for the main part do they depend on the market portfolio identification problem. Roll's conclusion (10) summarizes his objections on these points:

> Deviations from the return/beta linearity relation are frequently linked with some other phenomenon. The validity of such linkages is criticised using the Jensen measure of portfolio performance as an example. If the 'market' proxy used in the calculations is exactly (not significantly different from) ex-post efficient, all of the individual Jensen performance measures gross of expenses will be identically (not significantly different from) zero. They can be (significantly) non-zero only if the proxy market portfolio is (significantly) not efficient. But if the proxy market portfolio is not efficient, what is the justification for using it as a benchmark in performance evaluation?[1]

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*We have benefitted greatly from the thoughtful comments and suggestions of L. Dann, M. Jensen, C. F. Lee, J. Long, R. Roll, C. Smith, R. Verrecchia and an anonymous referee. Responsibility for the content of the paper, of course, is ours.

[1]Roll [1977, p. 132].

This statement is explicitly critical of the use of the security market line as a benchmark for asset or portfolio performance. If one agrees with the point being made in the rhetorical question, that the benchmark portfolio should be efficient, then the statement is strongly critical. It effectively eliminates the usefulness of the security market line as a benchmark. In a companion piece Roll [1978] criticizes in detail the accepted methodology of portfolio evaluation.

Our discussion has focused on Roll's conclusion (10). This is justifiable because (a) most of what we have to say concerns this conclusion and because (b) this conclusion is the most damning. It is the most damning because it implies that the theory has little operational usefulness, even if the theory is valid.

Empirically validating any economic theory is a difficult task. And it is in this area that Roll's [1977] contribution must be considered as paramount. His analysis is primarily concerned with problems associated with the testability of the theory. His conclusions based squarely on the mathematics of the efficient set are unassailable. However, his conclusion (4) we consider severe:

> The theory is not testable unless the exact composition of the true market portfolio is known and used in the tests. This implies that the theory is not testable unless all individual assets are included in the sample.[1]

The importance of Roll's criticism should be self-evident. Few economic theories have attained the level of operational elegance achieved by the Sharpe [1964], Lintner [1965] and Black [1972] models of capital asset pricing. The theory has been directly applied in a large number

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[1] Roll [1977, p. 130].

and variety of empirical studies. Tests of capital market efficiency, the effects of information events on share prices, the performance of mutual fund managers and the efficiency of other markets (e.g., merger studies) are a sample of the variety of such studies. In addition, there has been a plethora of related studies, for example, studies concerned with the problems of measuring the asset pricing model's implied measure of risk. One merely has to skim the title pages of the finance journals over the past decade to become impressed by the central position assumed by this model in the field of finance.

Our analysis is primarily concerned with the appropriateness of using security market line for the purpose of evaluating portfolio performance. Our approach is choice and information theoretic within the framework of a general equilibrium capital asset pricing model. Our analysis is unique in this respect. We define an individual as superior with respect to the information he holds. We then query whether a security market line (SML) analysis will correctly designate him as superior. This is opposed to earlier analyses that define superiority with respect to a given benchmark.[1] In other words we try to answer the question concerning the appropriateness of the benchmark.

Section II that follows contains a synthesis and discussion of Roll's analysis. Our discussion introduces our primary objection--that Roll does not allow for the possibility of superior performance. Section III contains our choice and information theoretic analysis. We demonstrate that the security market line can be a useful tool for detecting superior

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[1]An exception is Jensen [1972].

performance. In Section IV we examine residual analysis and conclude that residual analysis is useful. In Section V we offer some thoughts with regard to the problems associated with joint hypotheses and proxies in asset pricing theory tests. We see little reason to either reject the theory of asset pricing or the information provided by tests of the theory solely on the basis of the existence of these problems. Section VI concludes our study.

II. A SYNTHESIS AND DISCUSSION OF ROLL'S ANALYSIS

Roll's [1977] conclusion (10) is based on several statements derived from the mathematics of the efficient set. To help understand Roll's conclusion and our analysis, the following selected statements from Roll [1978] are repeated.[1]

S3: If the selected index is mean/variance efficient, then the betas of all assets are related to their mean returns by the same linear function.

S4: If, for some selected index, the betas of all assets are related to their mean returns by the same linear function, then that index is mean variance efficient.

S6: For every ranking of performance obtained with a mean/variance non-efficient index, there exists another non-efficient index which reverses the ranking.

These three statements are sufficient for an understanding of the analysis leading to Roll's conclusion (10). The statements are true for ex ante probability assessments as well as for ex post sample statistics. Thus, for example, if the selected index happened to be ex post mean/variance efficient,[2] all individual assets and portfolios would plot exactly on the security market line derived from the selected index (S3).[3]

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[1] These statements are derived from the efficient set mathematics as outlined in Roll [1977]. The derived statements in Roll [1978], however, relate more directly to the issue at hand.

[2] An index is ex post mean/variance efficient if it lies on the sample efficient frontier derived from the realized average return vector and sample covariance matrix. Why anybody would ever want to use such an index is not obvious. The concept of ex post efficiency is without economic content and the probability of an ex ante efficient portfolio ever being ex post efficient most certainly would be small (what statisticians refer to as a set of measure zero).

[3] This assumes that portfolio returns are measured gross of any expenses generated by the portfolio manager.

No portfolio or asset could indicate either superior or inferior performance using the securities market line criterion. If the selected index happened to be ex ante efficient, there could be deviations from the ex post security market line, but they would be statistically insignificant and would tend to disappear in repeated sampling over many intervals (S3).[1] Statements S4 and S6 tell us that if we do find significant deviations from the security market line, the selected index is not mean/variance efficient and the deviation can tell us nothing about the relative performance of portfolios or assets. These results are true whether the portfolios under evaluation are constant composition portfolios or portfolios of changing composition across time. That is, as long as the selected index is ex ante mean/variance efficient for all time intervals under consideration S3 will rule whether the portfolios are of constant or changing composition.

The above would appear to leave little room for positive statements to be made concerning portfolio or asset performance; this is the point of Roll's conclusion (10). For example, if in a securities market line analysis one discovers significant deviations from the empirical line, what can be said? One can say (for sure) that the index was not mean/variance efficient (from S3), and S6 implies that any statements about performance are questionable. On the other hand, a finding of no significant deviations in an analysis is consistent with the index having been mean/variance efficient in which case no portfolio manager or

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[1] See footnote 3 on previous page.

investment advisor could have consistently picked winners relative to the benchmark. This finding would be consistent with the simultaneous hypotheses of capital market efficiency and the index being mean/variance efficient,[1] but no positive statement about performance can be made.

Thus, Roll's conclusion (10) concerning portfolio performance appears sustained. Roll [1978, p. 1060] amplifies this conclusion by stating:

> Individual differences in portfolio selection ability cannot be measured by the securities market line criterion. This was the general thrust of the preceding argument. If the index is ex ante mean/variance efficient, the criterion will be unable to discriminate between winners and losers. If the index is not ex ante efficient, the criterion will designate winners and losers; but another index could cause the criterion to designate different winners and losers and there is no objective way to ascertain which index is correct.[2]

We claim the contrary; that individual differences in portfolio selection ability can be measured by the securities market line criterion. However, in order to be able to detect superior performance, the possibility of such must be allowed. By assuming that the selected index is mean/variance efficient, with respect to the correctly assessed efficient frontier, Roll assumes away the possibility of superior performance.[3] He is certainly right that, when such an index is used as a benchmark, superior

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[1]Capital Market efficiency refers to efficiency in an information sense as opposed to a mean/variance sense.

[2]By individual differences, we presume Roll means simply whether an individual is designated as a winner or a loser. We do not address the issue of whether meaningful comparisons between winners, for example, can be made. There may be a problem associated with comparing performance between managers of portfolios with different risk.

[3]The correctly assessed efficient frontier is derived using the probability density functions of nature's irreducible stochastic elements. Thus, the correctly assessed efficient frontier correctly captures all information that is knowable.

performance will be undetected. Under this assumption, superior performance is impossible. This benchmark is efficient with respect to all that is knowable--all probability distributions are correctly assessed—and consequently there can be no superior performance relative to this benchmark.

An interpretation of mean/variance efficiency that allows superior performance is as follows. We posit that the benchmark index is mean/variance efficient relative to the joint probability density function of returns assessed by the "market." We use the term "market" metaphorically as in Fama [1976, p. 168]:

> Thus, in deriving testable implications of the hypothesis that the capital market is efficient, we structure the world in terms of a 'market' that assesses probability distributions on future prices and then sets current prices on the basis of these assessed distributions. Strictly speaking, this implies that investors have monolithic opinions about available information and act single-mindedly to ensure that their assessments are properly reflected in current prices. What we really have in mind, however, is a market where there is indeed disagreement among investors but where the force of common judgments is sufficient to produce an orderly adjustment of prices to new information.

If the capital market is inefficient in processing information, the joint probability density function of returns assessed by the market may differ from the correct joint probability density function and significant deviations from the empirical security market line may be observed. Of course, if this is the case our index is not mean/variance efficient with respect to the correctly assessed efficient frontier. However, with this interpretation relative to the index we can designate winners in the sense that those managers or advisers who plot deviations significantly above the line are superior relative to "the market." Of course, if we find no significant deviations, the simultaneous hypotheses of the capital market being efficient in information processing and the mean variance efficiency of the index relative to market beliefs are sustained.[1]

Another issue raised in the Roll [1978, p. 1060] quote above is concerned with the sensitivity of ranking to the choice of index. This issue follows directly from statement 6. It is true that choosing an index that is inefficient relative to the markets assessed efficient frontier can yield incorrect performance evaluations and that another similarly inefficient index could reverse the rankings. The task of choosing an index that is mean/variance efficient relative to the appropriate set of beliefs is an important task. The capital asset pricing model suggests that the value-weighted market portfolio should be such an index. We postpone until section V our discussion of problems associated with obtaining a good measure of the returns on the market portfolio.

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[1] "This is the rub in tests of market efficiency. Any test is simultaneously a test of efficiency and of assumptions about the characteristics of market equilibrium." Fama [1976, p. 137].

III. PORTFOLIO PERFORMANCE

In this section we formalize our discussion and demonstrate that the security market line can be a useful tool in detecting superior portfolio performance. To do this, we presume an individual with better information than the market about some aspect of security returns and show he will on average plot above the ex post security market line (SML). We are unable to show this superior designation under completely general information, thus the logical possibility of incorrect designation by the SML remains. The conditions needed for proving correct designations are relatively weak, however.

Previous analyses have assumed that a superior portfolio manager plots above the security market line. In fact superiority has been defined, "...as a portfolio whose returns are consistently greater than those implied by its level of systematic risk."[1] Roll's analysis calls into question the assumption that superior performance implies and is implied by plotting above the security market line. We define superiority in an information sense. One individual is superior if he has better information than others. The question we then address is whether an individual who meets our definition of superiority will be so designated in an SML analysis.

A one-to-one correspondence in winner/loser designations between information superiority and SML superiority would, of course, validate the SML criterion. Less than a one-to-one correspondence does not imply we should reject the criterion, however. That is, if the SML is useful

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[1] Jensen [1969, p. 192].

in showing superior performance under a wide variety of, but not all conditions, the criterion can still be useful unless a better criterion can be shown to exist.

At this point some discussion on the philosophy of the analysis to follow is in order. We would like to have a completely general equilibrium framework for analysis. The capital asset pricing model, for example, is a general equilibrium model, but this framework assumes homogeneous beliefs among market participants. Thus, we have a problem if we assume a capital asset pricing model general equilibrium and we allow an individual to have better information than that implied by the homogeneous beliefs that establish the equilibrium. One way around this problem is to assume a general equilibrium as used by Hirschleifer [1975]. He assumes an economy that is dominated by agents who share homogeneous beliefs and an individual with deviant beliefs. The deviant belief individual is assumed to have essentially "zero weight" in the economy. In reality, of course, if one performs an SML analysis and detects superior performance, the assumed general equilibrium model (CAPM) cannot literally hold. However, by attributing very little weight to the deviant individual, it can hold approximately.[1,2]

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[1] Another way around the homogeneous belief problem is through the concept of "consensus beliefs." This alterntive scenario allows for widely diverse, heterogeneous beliefs on the part of asset traders. While the beliefs are heterogeneous, the arrived-at prices represent some sort of consensus of the market participants. That is, there is some probability belief, which, if held by all traders, would result in the same prices as those currently existing. This no-price-difference probability belief can serve as the market assessment, if this probability distribution is multivariate normal or if all investors have quadratic utility (more important in the finite states of the world model), prices are those arrived at in the CAPM with homogeneous probability beliefs. The existence of a "consensus belief" has been formally demonstrated by Verrecchia [1978].

[2] Thus, we've outlined a scenario wherein all individuals would not hold the levered market portfolio, but assets would be priced consistent with the CAPM. The pricing result is the important implication of the CAPM. The implications of the CAPM that all individuals hold levered market portfolios is not important, nor is it consistent with the evidence anyway.

The remainder of this section is divided into two parts, each considering a different level of information a portfolio manager might have. Roll [1978] identifies two types of relevant information: (1) privileged information about specific securities, or (2) information on general market conditions. In the following part of this section we consider information of type (1). In the second part we consider the general case, where the portfolio manager has information about both individual securities and the market return.

III.1 A Model with Informed and Uninformed Individuals: Security Specific Information

Our model assumes an individual with better assessments than the market about specific securities. We assume all individuals assess the mean and variance of the market return identically. In particular, our informed individual has access to an "information service" which gives him one of L messages each period. After receiving any particular message $j = 1, \ldots, L$, we represent the informed individual's beliefs as a vector

$$\pi^{Ij} = (\pi_1^{Ij}, \ldots, \pi_N^{Ij}), \qquad \sum_{s=1}^{N} \pi_s^{Ij} = 1$$

which represents the probabilities he assigns to the N possible states of the world. We represent the market probability assessments (uninformed) as[1]

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[1]We are here assuming that the market (π^U) assessment remains stationary over time. This is similar to the assumptions of most of the econometric work on the CAPM.

$$\pi^U = (\pi_1^U, \ldots, \pi_N^U), \quad \sum_{s=1}^{N} \pi_s^U = 1.$$ [1]

We can thus define any message j by its vector

$$\eta^j = \pi^{Ij} - \pi^U \qquad \sum_{s=1}^{N} \eta_s^j = 0$$

which represents the changes in probability assessments associated with message j.

For our informed individual, there is also a probability distribution for receipt of the various messages.[2] We represent this as the vector

$$q = (q_1, \ldots q_L) \qquad \sum_{j=1}^{L} q_j = 1,$$

where q_j is the probability of receiving message j. We further assume that given any j,

$\eta^j \neq 0$ but $E^{Ij}(R_M) = E^U(R_M)$, where E^{Ij} is the expectations operator on probability beliefs π^{Ij}, E^U is the same operator on π^U, and R_M is the market rate of return. This assumption rules out the possibility of no information while maintaining our assumption of security information only (no market information).[3]

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[1] The informed individual is our "zero weight" deviant, thus prices are as if the uninformed beliefs are homogeneous.

[2] We assume, as in footnote 1 previous page, that this distribution is stationary over time.

[3] This is somewhat stronger than the assumption we need here, which is that $\eta^j \neq 0$ for some j where $q_j > 0$. We make this assumption for analytical and expository convenience.

We also assume that the informed and the uninformed have rational expectations. This rational expectations assumption is clear in the context of this model. That is, since the market never receives message j, each period it maintains the prior probability assessment π^U. Given message j, the informed individual has probability assessment $\pi^{Ij} \neq \pi^U$. Yet both expectations can be rational, given the information available to each agent if and only if

$$\text{(A)} \quad \sum_{j=1}^{N} q_j \eta_S^j = 0 \quad \text{for every } S.$$

For if the informed expectations were "fulfilled," the unconditional (on j) probability distribution would have to be

$$\text{(B)} \quad \sum_{j=1}^{L} q_j \pi^{Ij} = \sum_{j=1}^{L} q_j(\pi^U + \eta^j) = \sum_{j=1}^{L} q_j \pi^U + \sum_{j=1}^{L} q_j \eta^j$$

$$= \pi^U + \sum_{j=1}^{L} q_j \eta^j.$$

From (B), it is clear that if (A) does not hold, π^U will not be the observed unconditional distribution. Similarly from (B), if (A) holds, π^U will be the observed unconditional distribution. Thus, we assume (A) holds and the informed expectations are fulfilled. People all have rational, but possibly different expectations.

We suppose that the market prices assets according to the capital asset pricing model (CAPM) of Sharp [1964], Lintner [1965], and Mossin [1966] in a complete market. We assume that investors have quadratic utility (consistent with the CAPM) so that they select mean/variance efficient portfolios relative to their probability beliefs.[1]

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[1] The quadratic utility assumption is used in the proof that follows in that it insures that any individual with beliefs π^U will optimally hold some combination of the market portfolio and the risk-free asset.

In a complete market, any portfolio K can be represented as a vector

$$Y_K = (Y_{K1}, \ldots, Y_{KN})$$

where Y_{Ks} represents the units of consumption received by holding portfolio K if state s occurs. For any individual i, we let U_{is} represent his utility conditional on state s occurring and U_i represent his unconditional utility function. We assume that utility is separable such that

$$U_i(C_i,Y_i) = U_i(C_i) + \sum_{s=1}^{N} \pi_s^i U_{is}(Y_{is}).$$

Here C_i measures i's current consumption and Y_i is the portfolio held by individual i.

Also in a complete market, the following conditions must hold:

a) $$V_r = \sum_s \omega_s X_{rs} \qquad \text{for all securities } r$$

b) $$\omega_s = \frac{\partial U_i(C_i,Y_i)}{\partial Y_{is}} \Big/ \frac{\partial U_i(C_i,Y_i)}{\partial C_i}$$

$$= \pi_s^i \frac{\partial U_{is}(Y_{is})}{\partial Y_{is}} \Big/ \frac{\partial U_i(C_i)}{\partial C_i} \qquad \text{for all individuals } i.$$

Condition a) states that the value of the rth security, V_r, is the sum of the priced payoffs, where ω_s is the price of a contingent claim on a unit of consumption in state s and X_{rs} is the total payoff (in units of consumption) of security r in state s. Condition b) notes that in equilibrium all individuals equate their marginal rates of substitution of future expected consumption for current consumption to the relevant state contingent claim prices.

Having completed our choice and information theoretic framework we now proceed with the analysis. Our concern is with the following theorem:

Theorem--If investor I's probability beliefs are correct, he will on average plot above the security market line as drawn by the uninformed investors. An equivalent way of stating this would be that investor I expects to plot above the security market line drawn by the uninformed investors. Symbolically, the statement of the theorem is

$$E^I(R_I) > R_F + [E^U(R_M) - R_F]\beta_I^U,$$

where $E^I(R_I)$ is the informed investor's unconditional expected return from his portfolio, R_F is the risk free rate, $E^U(R_M)$ is the uninformed's expectation of the return on the market portfolio and β_I^U is the $\text{cov}(R_I,R_M)/\text{var}(R_M)$ as assessed by the uninformed.

Proof--We first wish to show that given any message j,

$$(1) \qquad E^{Ij}(R_I) > R_F + [E^U(R_M) - R_F]\beta_{Ij}^U,$$

where the Ij superscript indicates the expectation is with respect to the probability assessment π^{Ij}. The U superscript likewise indicate expectations with probability assessments π^U. This condition has an obvious relationship to the security market line.

The formal proof of condition (1) is in the Appendix. The condition is established by constructing a hypothetical uninformed individual with the same utilty function and current consumption choice as our informed individual. We show the informed individual buys more future consumption than does the uninformed in states where his probability assessments are larger than are the uninformed's ($\eta_s^j = \pi_s^{Ij} - \pi_s^U > 0$). Likewise he buys less than the uninformed in states where his probability assessment is less. We then show that the informed individual, because his probabilities are correct, earns a higher average return than the uninformed expects him to. Since the uninformed individual expects him to plot directly on the security market line, using the uninformed estimate of beta, the informed individual will beat the security market line.

Condition (1) is a conditional expectation and the theorem is concerned with the unconditional expectation. The unconditional expected return of the informed's portfolio, $E^I(R_I)$, is obtained by averaging condition (1) over messages:

$$E^I(R_I) = \sum_{j=1}^{L} q_j \, E^{Ij}(R_I) > R_F + [E^U(R_M) - R_F] \sum_{j=1}^{L} q_j \, \beta_{Ij}^U . \tag{2}$$

This proves the theorem if the uninformed investors estimate the portfolio beta of the informed individual as the average of his single period portfolio betas.

<u>The Time Series Estimation Problem</u>. In fact, however, the uninformed investor is likely to use econometric techniques to measure the informed individual's beta. We now proceed to prove the theorem if this econometrically estimated ex post beta, β_I^E, is used by the uninformed investor.

The econometrically estimated beta will be

$$\begin{aligned} \beta_I^E &= \frac{COV(R_I, R_M)}{\sigma^2(R_M)} \\ &= \sum_j \frac{q_j [E^{Ij}(R_{Ij} R_M) - E^{Ij}(R_{Ij}) E^U(R_M)]}{\sigma_U^2(R_M)} \end{aligned} \tag{3}$$

using the rational expectations assumption, where $\sigma_U^2(R_M)$ is the variance of the market return assessed with the uninformed beliefs. Since the informed individual has no information about the market, his expectation of the market return and the variance of the return assessed with his beliefs will be the same as the uninformed mean and variance. This reduces equation (3) to

$$\text{(4)} \qquad \beta_I^E = \sum_j q_j \, \beta_{Ij}^I \ .$$

A necessary assumption for the proof is that

$$\sum_{j=1}^{L} q_j \, \beta_{Ij}^U = \sum_{j=1}^{L} q_j \, \beta_{Ij}^I \ .$$

Here, β_{Ij}^U is the covariance of the informed individual's portfolio return with the market return divided by the variance of the market return, assessed with the (uninformed) π^U distribution. β_{Ij}^I is the same ratio assessed with the (informed) π^{Ij} expectations. This assumption is then that the average beta of the informed individual's portfolio is the same whether assessed with the informed or uninformed expectations. The assumption will hold when information is in some sense "symmetric" with respect to market returns; i.e., there is as much information (η_s deviation) about high market return states as about low market return states. We will discuss the effects of relaxing this assumption later in this section.

Thus, we assume that the average beta using the informed's expectations will be equal to the average beta using uninformed expectations. This, combined with equation (4), establishes that

$$\beta_I^E = \sum_j q_j \, \beta_{Ij}^U, \text{ and thus}$$

(5)

$$E^I(R_I) > R_F + [E^U(R_M) - R_F]\beta_I^E$$

from (2).

This proves the theorem in the case where the uninformed investors use time series regression techniques to estimate the informed investor's portfolio beta.

We can now discuss the effects of relaxing the assumption that the average β_{Ij}^U must equal the average β_{Ij}^I. Without this condition, the above proof will not go through and we can not generally support the standard security market line analysis using an econometric beta.[1] However, a modification of the econometric technique can still be used to make unambiguous statements about portfolio performance.

That is, instead of estimating the beta of the portfolio of the informed individual through time series analysis, we can use the betas of the individual securities composing his portfolio by weighting these individual security betas by the informed individual's portfolio weights. In a particular period, we arrive at an estimate of β_{Ij}^U for that period.

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[1] The condition will not hold for all information possibilities. In particular, consider a situation where the information in message j is only about favorable states of the world; that is, $\eta_s^j \neq 0$ only when $X_s > E^U(X_s)$. Here, the informed individual's expectation of his return will be greater than the market's expectation of his return only when the market return is greater than average. This must result in β_{Ij}^I being greater than β_{Ij}^U. If this condition holds for all messages, then the average β_{Ij}^I must be greater than the average β_{Ij}^U for this individual.

Averaging these single period estimates over time, we arrive at an estimate of $\sum_j q_j \beta^U_{Ij}$. From equation (2), we know that if we use this average β^U_{Ij} in our portfolio performance test, we are able to detect superior performance for the case of security specific information. Thus, using individual security betas and averaging as described above, we can obtain a beta that validates the security market line criterion.

III.2 General Information

Thus, with security specific information the informed investor will be correctly designated in a security market line analysis. We now discuss the problem in the context of general information. Hence, we remove the restriction that $E^{Ij}(R_M) = E^U(R_M)$. As stated at the beginning of this section, with completely general information the possibility of incorrect designation by the SML remains. In the remainder of this section we derive a sufficient condition for an informed individual, with general information, to be correctly designated in an SML analysis. We also relate our analysis to a similar analysis done by Jensen [1972]. Jensen's analysis refers directly to a potential econometric problem that arises in the framework of our analysis.

As shown in the Appendix, if the informed individual's consumption and wealth are constant across messages, condition (2) will hold:[1]

[1] In our scenario messages are received across time. Consequently, we are assuming the informed's consumption and wealth are time independent as well as message constant. These assumptions seem to fade from reality, but the reader should keep in mind that the assumptions are sufficient for condition (2) to hold.

$$(2) \qquad E^{I}(R_I) = \sum_{j=1}^{L} q_j \, E^{Ij}(R_I) > R_F + [E^U(R_M) - R_F] \sum_{j=1}^{L} q_j \, \beta^U_{Ij}.$$

This proves the theorem if the uninformed investors estimate the portfolio beta of the informed individual as the average of his single period betas. Thus, the sufficient condition for validating the security market line benchmark, if the beta of the portfolio is measured as the average single period uninformed beta, is that the portfolio manager with constant wealth selects the same consumption regardless of the message.

The Time Series Estimation Problem. In this general case, however, we have a very difficult time extending this proof to include the SML drawn with the econometrically estimated beta. Where information was security specific we were able to demonstrate the plausibility of a correspondence between the econometrically estimated beta, β^E_I, and the average of the single period uninformed betas. With general information this correspondence is less plausible. For example, equation (3),

$$(3) \qquad \beta^E_I = \sum_j \frac{q_j [E^{Ij}(R_{Ij} R_M) - E^{Ij}(R_{Ij}) E^U(R_M)]}{\sigma^2_U(R_M)}$$

can no longer be interpreted as a weighted average of any one individual's assessments.[1] However, Jensen's [1972] analysis indicates this econometric problem is of little empirical relevance.

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[1]With $E^{Ij}(R_M) = E^U(R_M)$ the numerator of the right hand side of equation (3) is a weighted average of $COV^{Ij}(R_{Ij}, R_M)$, but with the general information assumption the right hand side includes uninformed expectations as well as informed.

Jensen's analysis of the time series estimation problem associated with assessing performance focuses initially on the situation where the informed investor has information about expected market conditions only. Under his assumptions, the optimal adjustment of a portfolio manager to information about the market return involves setting

$$\beta_{It} = \beta_I + \Theta_I \pi_t'$$

where $\Theta_I (> 0)$ is a parameter that depends on the risk aversion of the informed individual and his confidence in his prediction of the market return, and π_t' is the difference between his expectation of the market return in period t and the uninformed expectation.[1] Since $E(\pi_t')$ is zero, β_I is the average beta of the informed investor's portfolio.

Jensen goes on to show that this optimal adjustment will normally result in an econometric overestimate of β_I. That is,[2]

$$\text{Plim } \hat{\beta}_I = \beta_I + \Theta_I \rho^2 E^U(R_M - R_F)$$

where ρ is the correlation between $E^I(R_M)$ and the observed R_M (across time). Thus, if the informed individual has any predictive ability ($\rho > 0$),

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[1]We have simplified Jensen's analysis for the purposes of our paper. In particular, we have assumed $\pi_t^* = \pi_t'$ in his terminology (or that the informed forecasts are "optimal") and have translated his terminology and symbols into our framework where possible. Also, note that Θ_I is assumed independent of t in this formulation. It is at this point that Jensen implicitly introduced the same assumptions that we do; i.e., those constraining the informed individual's wealth and consumption to be the same across time.

[2]We assume here and in what follows that $E(\pi')^3 = 0$, or that the forecasts are "symmetric" about $E(R_M)$. This makes the exposition and results much simpler.

his beta estimate will be biased upward; only where he has no ability whatsoever ($\rho = 0$) will the econometric estimate of beta be unbiased.

This problem of bias in the beta coefficient has consequences for portfolio performance evaluation. Again, Jensen shows that the manager's ability to predict will result in an econometric estimate of his distance above the security market line as

$$\text{Plim } \hat{\delta} = \delta_I + \Theta_I \rho^2 \sigma^2(R_M) - \Theta_I \rho^2 [E^U(R_M - R_F)]^2$$

where δ_I is a measure of the superior performance due to the informed individual's ability to predict returns on specific securities. Jensen goes on to show that the second term, $\Theta_I \rho^2 \sigma^2(R_M)$, is the correct measurement of the excess return the portfolio manager receives for his ability to forecast the market. Thus, the informed manager's estimated performance will be biased downward by the last term and any superiority will be understated. In fact, if $\sigma(R_M) < E^U(R_M - R_F)$, the portfolio manager would show up econometrically as having worse performance than if he could not predict the market return at all! However, from an empirical standpoint this possibility is uninteresting.[1]

What then have we been able to show? For a broad classification of information, security specific, we have shown a one-to-one correspondence between information superiority and SML designation. This one-to-one

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[1]Fisher and Lorie [1970] report the post World War II (1945-1965) standard deviation of annual rates of return on the arithmetic index of NYSE stocks as .197. They also reported the average annual rate of return on this index over the same period as .138. If the average riskless rate was above .04 over this period then an estimate is that $\sigma(R_M) \cong 2E^U(R_M - R_F)$.

correspondence can obtain using standard time series estimation procedures. We have also shown sufficient conditions for correct designation in the general information case. Jensen's analysis indicates that, despite a downward bias, standard time series estimation procedures yield the correct designation here as well. The possibility of incorrect designation in the general information case remains. However, the sufficient conditions for correct designation are not so strong as to reject the SML criterion or its usefulness for the general information case. Our analysis does not indicate that inferior managers will be incorrectly designated;[1] only that a possibility of incorrectly designating a superior manager exists.

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[1]An inferior manager would be one with no information advantage who generates expenses, or one whose information advantage is outweighed by the generated expenses.

IV. ON RESIDUAL ANALYSIS

As mentioned in the introduction, Roll's conclusion (10) can be easily interpreted as being critical of the empirical methodology known as residual analysis. In this section we address the issue of the usefulness of residual analysis as an empirical tool. We find, first, that only a very special, inappropriate index would be unable, as Roll suggests, to pick up deviations from the security market line. Second, we show that residuals computed against the market index will be appropriate if assets are priced according to the capital asset pricing model.

Residual analysis is an empirical methodology designed to measure the effects of information events on security prices. The effects are measured by comparing a security's return when the information event occurs to the ex ante expected return. As such, the problem of residual analysis differs from the problem of portfolio evaluation in an important way. Portfolios of securities selected by investment managers are selected ex ante. Presumably the managers have expectations of superior future performance. Contrarily, residual analysis involves an ex post selection rule. That is, a security is picked for a residual analysis "portfolio" because some event specific to that security has occurred and the desire is to determine the effect of the event on the price of the security. Thus, even if the index used were mean-variance efficient with respect to all information available at the beginning of the period, positive and negative deviations would be detected by a security market line type residual analysis. It appears that only if the index were ex post efficient in a very special way, as we shall explain below, would these deviations be undetected.

Looking at the specifics of residual analysis should help clarify this point. Let us examine, then, the two factor market model type of residual analysis as performed by Mandelker [1974], Jaffe [1974] and others. This methodology requires the estimation of the security market line for each date residuals are desired. Historically, this estimation has been accomplished by regressing a portfolio return vector on a vector of the associated portfolio betas for each date.[1] The estimated regression coefficients are used as the parameters of the security market lines. To perform the residual analysis betas are calculated for each security in the residual analysis portfolio. Then residuals are estimated using the calculated betas, the securities' returns and the regression coefficients. If $\hat{\gamma}_{ot}$ and $\hat{\gamma}_{1t}$ are the estimated regression coefficients for a particular date, a residual, $\hat{e}_{rt}$, is estimated for a security r in the portfolio as

$$\hat{e}_{rt} = R_{rt} - \hat{\gamma}_{ot} - \hat{\gamma}_{1t}\,\hat{\beta}_r,$$

where R_{rt} is the realized return of security r in period t and $\hat{\beta}_r$ is the estimated beta from the previous period.

Roll's mathematics tells us that all deviations from the security market line are zero if an efficient portfolio is used as the index. The required index need be efficient relative to the observed expected return vector and the same variance-covariance matrix used in computing beta.

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[1] See Fama and MacBeth [1973] for the details. The portfolio return vectors are from the dates of interest, and the vectors of portfolio betas are always estimated from sample periods prior to the dates of interest.

In residual analysis, returns for <u>specific</u> time periods are used and betas are not calculated over these same time periods. Thus, to apply mathematical identities to get zero residuals in the analysis described above, one must employ an index that is mean/variance efficient

(1) treating the realized return vector for period t as the relevant average return vector; and

(2) treating the sample covariance matrix of the previous period (on which betas are measured) as the relevant covariance matrix.[1]

Such an index would thus involve a special form of ex post efficiency.

Why one would advocate the use of this index is unclear. As stated earlier, the security market line is used in residual analysis to compute the <u>ex ante</u> expected return on a security (before an information event). Using any type of <u>ex post</u> efficient index will be inappropriate for yielding this ex ante expected return.

Having established our first conclusion of this section we now proceed to show that residual analysis, using the market portfolio as the index, is valid. This, of course, is contrary to the criticism implied by Roll's conclusion (10).[2]

Our model is essentially that of the last section. We assume that all individuals in the economy are initially uninformed. We assume further that the market prices assets according to the capital asset pricing model

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[1] Furthermore, this same covariance matrix must be used in calculation of the betas that are put into the cross sectional regression yielding $\hat{\gamma}_{ot}$ and $\hat{\gamma}_{it}$.

[2] Roll's most recent paper [1978] does recognize that residual analysis is valid under certain conditions.

and efficiently processes all information available at the beginning of the period. The expected return-risk relationship is thus

$$E^U(R_r) = \frac{\sum_s \pi_s^U X_{rs}}{\sum_s \omega_s X_{rs}} - 1 = R_F + [E^U(R_M) - R_F]\,\beta_r^U .$$

where R_r is the return on security r, X_{rs} is the payoff of security r in state s, and β_r^U is the market's assessment of the beta of security r. With this scenario the market portfolio is truly ex ante efficient. During the period, an information event occurs that changes the correct probability distribution to π_s^I. Residual analysis is concerned with firm specific information, hence we retain the assumptions that $E^I(R_M) = E^U(R_M)$ and that $\sigma_I^2(R_M) = \sigma_U^2(R_M)$.

Let us now examine the residuals we should expect in this model using the market portfolio as the index. By arguing as in section III.1, the expected residual, e_{rt}, will be[1]

$$E^I(e_{rt}) = E^I(R_{rt}) - R_F - [E^U(R_M) - R_F]\,\beta_r^U$$

$$= \frac{\sum_s (\pi_s^I - \pi_s^U) X_{rs}}{V_r} ,$$

where V_r is the value of security r. This equation seems to give us exactly what we desire in a residual, the difference between the expected return given the information event $\frac{(\sum_s \pi_s^I X_{rs})}{V_r}$ and the ex ante expected return $\frac{(\sum_s \pi_s^U X_{rs})}{V_r}$.

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[1]See the proof in the Appendix to section III.A for the development of the formula.

Additional intuition about the residual can be gained by rewriting our last equation as

$$E^I(e_{rt}) = \frac{1}{V_r} \operatorname{cov*}(\eta_s, X_{rs})$$

where cov* indicates a sample covariance and η_s represents, as in section III, the change in probability of state s due to the information event. This residual will then be expected to be positive or negative as this sample covariance is positive or negative.

Notice now that the covariance will be positive for favorable information and negative for unfavorable information. That is, favorable information should show increases in the probabilities (high η_s) of high payoff states for the security (high X_{rs}); it should also show decreases in the probabilities (low η_s) of low payoff states (low X_{rs}). This should clearly result in a positive covariance in the above equation, if the information is favorable. One can show, analogously, that unfavorable information will result in a negative covariance.

Thus, if assets are priced as in the capital asset pricing model, residual analysis will measure the effects desired. A favorable information event for security r will generally be associated with positive residuals for that security; an unfavorable information event will generally be associated with negative residuals. Furthermore, the expected residual for a period with an information event will be the change in the expected return caused by the new information.

V. ON TESTS OF THE ASSET PRICING MODEL

At this point in the paper we have accomplished our primary intentions, and we now turn to Roll's [1977] conclusion (4) which we consider severe. Roll preludes his [1977] study with the following:

> If the horn honks and the mechanic concludes that the whole electrical system is working, he is in deep trouble ...
> Pirsig (1974)

As a prelude to this section we offer the conjecture that the honking horn does provide some information and that if the mechanic is prohibited from further tests, he ought not ignore the information provided by the proxy.

Roll [1977] identifies three problems with tests of the capital asset pricing model:[1]

a) The only testable implication of the model is that the true market portfolio is mean-variance efficient.

b) The return of the true market portfolio is not used in any of the empirical tests to date and is virtually impossible to measure. The theory is not testable unless the exact value-weighted market portfolio of all assets is used.

c) All tests of the model involve joint hypotheses, one of which is that the market portfolio is correctly measured. Since we know the market portfolio is not correctly measured, the rejection of the joint hypothesis tells us very little.

We sympathize with all 3 problems and believe they are all of some importance. Criticism (a), especially, seems to have cleared up some of

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[1] This is our summary of his criticisms, which he states somewhat differently.

the confusion involved in earlier tests. However, criticisms (b) and (c) impose extremely severe criteria on empirical work that few, if any, econometric studies can meet.

Criticism (b) seems to say that no proxy for the market return will suffice for testing the theory. Roll correctly suggests that the use of a proxy variable increases the risk of type I and type II error in testing. Yet, it is almost impossible to find the "true" measure of any variable in economics. Proxies must be used constantly to test all types of economic theories. Are we to abandon studies of inflation because the change in the Consumer Price Index (CPI) is merely a proxy for the inflation rate? Are we to abandon cencentration studies because the 4-firm concentration ratio is merely a proxy for the concentration of an industry? Are we to abandon all empirical studies?

Clearly, Roll finds something especially worrisome about proxies for the return on the market portfolio. But the only "special" problem he discusses[1] is that highly correlated proxies can yield different conclusions about the validity of the CAPM. This problem is again, however, not unique to CAPM tests. Consider, for example, the testing of a theory which explains some variable Y as a function of inflation. Suppose the following regression is run over N periods,

$$Y_t = a + b(\Delta CPI_t) + u_t$$

where ΔCPI is used as a proxy for inflation. Suppose further that the

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[1]Roll also discusses an econometric aggregation problem in the portfolio grouping technique, but this is not a proxy problem. The aggregation problem is again a familiar one to econometricians.

results offer some support for the theory, with a significant t-statistic on b and $R^2 = .25$. Now, by the geometry of the situation, there must be some other proxy for inflation, correlated .866 with ΔCPI, which shows no relationship <u>at all</u> to Y ($R^2 = .00$).[1] This proxy can also be constructed simply by assigning different weights to the price series that compose the CPI, if we have N linearly independent price change vectors for individual goods.[2] It seems that this proxy and the many others like it have exactly the same problem that Roll finds so worrisome in the market proxy.

Problem (c) is also a common one in economic hypotheses. That is, virtually all tests of economic models involve joint hypotheses. The simple supply-demand model, for example, is not independently testable; it must be tested in conjunction with some hypothesis about how one of the curves has shifted. Other models are tested jointly with, if nothing else, the hypothesis that the data and variables used correspond to the theory presented. In fact, linking the CAPM with this last hypothesis is the joint hypothesis Roll finds so troublesome.

What we have done so far is to argue that the problems Roll sees in the CAPM tests are also found in other econometric studies. Unfortunately, this is not sufficient to refute, or even really minimize, his criticisms. In fact, he is certainly correct in that a <u>definitive</u> test of the theory

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[1]Since R = .5 is the cosine of the angle between the Y vector and the ΔCPI vector in N-space, the angle between these two must be 60°. Thus, there is some vector, X, in the same plane as Y and ΔCPI that is 30° from ΔCPI and 90° from Y. The correlation between X and ΔCPI will be $\cos 30^\circ = .866$; the correlation between X and Y will be $\cos 90^\circ = 0$.

[2]It may be that some goods would have to be given negative weights, but this is the same problem Roll mentions in his criticism of Black, Jensen, & Scholes.

cannot be made without solving these problems. It is also unfortunately true, however, that definitive tests are nearly always impossible.

We disagree with Roll in his almost total condemnation of all empirical studies to date, implying that they provide virtually <u>no</u> information at all. We believe there is some information in these tests, even with imperfect proxies testing joint hypotheses. More importantly, this information is the <u>best</u> available. It does no good to ignore this information without providing some better information in its place.[1] In an ideal world, these problems would not exist—and we would certainly support the creation of such a world, were it costless--but this provides little justification for rejecting (ignoring) studies done in the world in which we now live.

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[1] In fairness to Roll, he is moving in this direction in Part II of his paper.

VI. SUMMARY AND CONCLUDING THOUGHTS

Our examination has been concerned with the empirical relevance of the capital asset pricing model. In particular, we have explored the worth of (1) tests of portfolio performance using the security market line benchmark, (2) tests of the effects of information events through residual analyses, and (3) tests of the CAPM itself. We conclude that, although there are potential problems with all three types of tests, they are valid tests.

Superior portfolio managers are reasonably detectable in a properly performed security market line analysis. Favorable and unfavorable information events will be similarly, on average, identified with positive and negative residuals. Thus, Roll's rhetorical question on the use of an index that is not "truly" efficient is answered. The appropriate index is one that is efficient relative to the "market" ex ante beliefs. If the market is not processing information efficiently, the appropriate index should not be truly efficient. Furthermore, the capital asset pricing model tells us that the value-weighted market portfolio is efficient relative to the market beliefs.

We sympathize with the criticisms offered by Roll concerning tests of the CAPM itself. However, we have argued that these criticisms can be leveled at virtually any econometric study. We conclude that the tests provide some information about the validity of the model and should not be dismissed out-of-hand.

Finally, Roll's criticism should not be interpreted as a rejection of the capital asset pricing model itself. One must replace this theory with an alternative theory that predicts better and/or is more useful in order to invalidate the CAPM. Even evidence itself is not enough to invalidate the theory. As Stigler [1966] puts it

> The answer is that it takes a theory to beat a theory: If there is a theory that is right 51 percent of the time, it will be used until a better one comes along.

In the case of the CAPM, it is far from clear what the suggested alternative is.

APPENDIX

Proof of Condition (1)

We restate here condition (1) and conditions a) and b):[1]

$$(1) \qquad E^{Ij}(R_I) > R_F + [E^U(R_M) - R_F]\, \beta^U_{Ij}$$

a) $V_r = \sum_s \omega_s X_{rs}$ for all securities r

b) $$\omega_s = \frac{\partial U_i(C_i, Y_i)}{\partial Y_{is}} \Big/ \frac{\partial U_i(C_i, Y_i)}{\partial C_i}$$

$$= \pi^i_s \frac{\partial U_{is}(Y_{is})}{\partial Y_{is}} \Big/ \frac{\partial U_i(C_i)}{\partial C_i} \qquad \text{for all individuals i.}$$

We represent I's optimal portfolio and current consumption, given message j, as $(Y^j_{I1}, \ldots, Y^j_{IN})$ and C^j_I respectively. Condition b) means that

$$\frac{\partial U_I(C^j_I)}{\partial C_I} = \frac{\pi^{Ij}_s}{\omega_s} \cdot \frac{\partial U_{Is}(Y^j_{Is})}{\partial Y_{Is}}, \text{ for all s.}$$

Now, let us construct a hypothetical individual U who holds beliefs π^U and has the same utility function as I. In general, for the same wealth as I, U will not select $C^j_U = C^j_I$. However, we select wealth for U such that $C^j_U = C^j_I$.

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[1]All notation is as defined in the text. To shorten this Appendix we refer to the informed individual as I and the uninformed as U.

For individual U also,

$$\frac{\partial U_U(C_U^j)}{\partial C_U} = \frac{\pi_s^U}{\omega_s} \cdot \frac{\partial U_{Us}(Y_{Us}^j)}{\partial Y_{Us}}, \text{ for all s.}$$

However, with our assumptions of identical utility functions and $C_U^j = C_I^j$:

$$\frac{\partial U_I(C_I^j)}{\partial C_I} = \frac{\pi_s^U}{\omega_s} \cdot \frac{\partial U_{Is}(Y_{Us}^j)}{\partial Y_{Us}} = \frac{\pi_s^{Ij}}{\omega_s} \cdot \frac{\partial U_{Is}(Y_{Is}^j)}{\partial Y_{Is}}$$

Using the last equality and rearranging yields

$$\frac{\dfrac{\partial U_{Is}(Y_{Is}^j)}{\partial Y_{Is}}}{\dfrac{\partial U_{Is}(Y_{Us}^j)}{\partial Y_{Us}}} = \frac{\pi_s^U}{\pi_s^{Ij}}$$

This result yields the following implications for quadratic utility (since it is concave):

$$\pi_s^U > \pi_s^{Ij} \rightarrow Y_{Us}^j > Y_{Is}^j$$

A1 $$\pi_s^U = \pi_s^{Ij} \rightarrow Y_{Us}^j = Y_{Is}^j$$

$$\pi_s^U < \pi_s^{Ij} \rightarrow Y_{Us}^j < Y_{Is}^j$$

This gives us the condition

A2 $$(\pi_s^{Ij} - \pi_s^U)(Y_{Is}^j - Y_{Us}^j) \geq 0, \text{ for all s}$$

And if $\pi_s^{Ij} \neq \pi_s^U$, $(\pi_s^{Ij} - \pi_s^U)(Y_{Is}^j - Y_{Us}^j) > 0.$

Individual I's expectation of his portfolio return is

$$\text{A3} \qquad E^{Ij}(R_I) = \frac{\sum_s \pi_s^{Ij} Y_{Is}^j}{\sum_s \omega_s Y_{Is}^j} - 1 = \frac{\sum_s (\pi_s^{Ij} - \pi_s^U) Y_{Is}^j}{\sum_s \omega_s Y_{Is}^j} + \frac{\sum_s \pi_s^U Y_{Is}^j}{\sum_s \omega_s Y_{Is}^j} - 1$$

Using the CAPM which we have assumed,

$$\text{A4} \qquad E^U(R_I) = \frac{\sum_s \pi_s^U Y_{Is}^j}{\sum_s \omega_s Y_{Is}^j} - 1 = R_F + [E^U(R_M) - R_F]\, \beta_{Ij}^U.$$

From equations A3 and A4

$$\text{A5} \qquad E^{Ij}(R_I) - (R_F + [E^U(R_M) - R_F]\beta_{Ij}^U) = \frac{\sum_s (\pi_s^{Ij} - \pi_s^U) Y_{Is}^j}{\sum_s \omega_s Y_{Is}^j}$$

$$= \frac{\sum_s (\pi_s^{Ij} - \pi_s^U)(Y_{Is}^j - Y_{Us}^j)}{\sum_s \omega_s Y_{Is}^j} + \frac{\sum_s (\pi_s^{Ij} - \pi_s^U) Y_{Us}^j}{\sum_s \omega_s Y_{Is}^j}.$$

We want to determine the sign of the right hand side. From CAPM theory, because U's expectations are equal to those of the market, U's portfolio must be a linear combination of the risk-free rate of return and the market portfolio, i.e., there exist α_1 and α_2 such that $Y_{Us}^j = \alpha_{Us}^j + \alpha_2^j X_s$ for all s where $X_s = \sum_r X_{rs}$ is the total consumption available in state s.

Hence,

$$\sum_s (\pi_s^{Ij} - \pi_s^U) Y_{Us}^j = \sum_s (\pi_s^{Ij} - \pi_s^U)(\alpha_1^j + \alpha_2^j X_s)$$

$$= \alpha_1^j \sum_s (\pi_s^{Ij} - \pi_s^U) + \alpha_2^j \sum_s (\pi_s^{Ij} - \pi_s^U) X_s .$$

Since $\sum_s \pi_s^{Ij} = 1 = \sum_s \pi_s^U$,

$$\sum_s (\pi_s^{Ij} - \pi_s^U) Y_{Us}^j = \alpha_2^j \sum_s \pi_s^{Ij} X_s - \alpha_2^j \sum_s \pi_s^U X_s$$

$$= \alpha_2^j (\sum_s \omega_s X_s)(E^{Ij}(R_M) + 1) - \alpha_2^j (\sum_s \omega_s X_s)(E^U(R_M) + 1)$$

$$= 0, \text{ since } E^{Ij}(R_M) = E^U(R_M) \text{ which we assumed.}$$

Hence, from equation A5

A6 $$E^{Ij}(R_I) - (R_F + [E^U(R_M) - R_F]\beta_{Ij}^U) = \sum_s \frac{(\pi_s^{Ij} - \pi_s^U)(Y_{Is}^j - Y_{Us}^j)}{\sum_s \omega_s Y_{Is}^j}$$

But, by condition A2 if $\pi_s^{Ij} \neq \pi_s^U$ for some s, the right hand side of A6 is greater than zero. Hence,

$$E^{Ij}(R_I) - (R_F + [E^U(R_M) - R_F]\beta_{Ij}^U) > 0$$

which establishes condition (1).

Demonstration of Condition (2) with General Information

Our assumptions are that the informed individual's consumption and wealth are constant across messages. Formally these assumptions imply:

$$C_I^j = C_I^*$$

and $$\sum_s \omega_s Y_{Is}^j = K = W - C_I^* .$$

Condition (2) is established by showing that

(2) $$E^{I}(R_I) = \sum_{j=1}^{L} q_j E^{Ij}(R_I) > R_f + [E^{U}(R_M) - R_F] \sum_{j=1}^{L} q_j \beta^{U}_{Ij}.$$

The right hand side of equation A5 above indicates this will be established if

A7 $$\sum_{j=1}^{L} q_j \frac{\sum_s (\pi_s^{Ij} - \pi_s^{U}) Y_{Us}^{j}}{\sum_s \omega_s Y_{Is}^{j}} \geq 0;$$

since the first term on the right hand side of equation A5 will be positive when weighted and summed over q_j by condition A2. By assumption $\sum_s \omega_s Y_{Is}^{j} = K$ and Y_{Us}^{j} is constant across j. Consequently, the numerator of A7 is important:

$$\sum_j q_j \sum_s (\pi_s^{Ij} - \pi_s^{U}) Y_{Us}^{j} = \sum_j q_j \sum_s (\pi_s^{Ij} - \pi_s^{U})(\alpha_1 + \alpha_2 X_s).$$

α_1 disappears as before and

$$\sum_j q_j \sum_s (\pi_s^{Ij} - \pi_s^{U}) Y_{Us}^{j} = \alpha_2 \sum_s \omega_s X_s [\sum_j q_j E^{Ij}(R_M) - E^{U}(R_M)].$$

Now, $\sum_j q_j E^{Ij}(R_M) = E^{U}(R_M)$ i.e., the unconditional probabilities of I and U are the same. Thus, condition (2) for this case is established.

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